Journal of Statistical Physics, Vol. 89, Nos. 5/6, 1997

Dobrushin States for Classical Spin Systems with Complex Interactions

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Received October 17, 1996; final Ausgust 8, 1997

We consider a classical spin system on the hypercubic lattice with a general interaction of the form

$$H=\frac{\beta}{4}\sum_{\substack{x,y:\\|x-y|=1}}|s_x-s_y|-h\sum_x x_x+\sum_A \lambda_A\prod_{y\in A}s_y$$

where $s_x \in \{-1,+1\}$ are the spin variables, β is the inverse temperature, h is the magnetic field, and λ_A are translation-invariant coupling constants satisfying $\lambda_A=0$ if diam $A>1$. No symmetry relating the configurations $s=\{s_x\}$ and $-s=\{-s_x\}$ is assumed. In dimension $d\geqslant 3$, we construct low-temperature states which break the translation invariance of the system by introducing so-called Dobrushin boundary conditions which force a horizontal interface into the system. In contrast to previous constructions, our methods work equally well for complex interactions, and should therefore be generalizable to quantum spin systems.

KEY WORDS: Dobrushin states; interfaces; surface tension; Pirogov–Sinai theory; complex interactions.

1. INTRODUCTION

In this paper, we discuss the construction of non-translation-invariant states for classical lattice systems which have contour representations without

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822/89/5-6-1

positive weights. Our principal motivation for studying such classical models is that the contour representations of many quantum systems have non-positive weights. In ref. [BCF96], we combine the methods of this paper with those of refs. [BKU96] and [DFF96] to construct non-translation-invariant states for certain quantum systems.

The methods of this paper rely heavily on the results of refs. [BF85], [HKZ88] and [BI92], which in turn are based on the classic works of Dobrushin [Dob72] and Gallavotti [Gal72], who developed convergent expansions for the Ising magnet at low temperatures: the Gallavotti expansion established instability of the interface in the two-dimensional Ising magnet, while the Dobrushin expansion established stability of the interface, and hence the existence of non-translation-invariant states, in the three-dimensional magnet. Briemont and Fröhlich [BF85] further developed interface and walk expansions, devising in particular expansions in terms of "decorated" interfaces. Holicky, Kotecký and Zahradnick [HKZ88] used Pirogov–Sinai theory [PS75] in the form developed in ref. [Zah84] to extend the Dobrushin analysis to models with no symmetry relating the two phases separated by the interface. Finally, Borgs and Imbrie [BI92] analyzed finite-size scaling effects in systems with dynamically generated interfaces with non-positive, but real contour weights, using their earlier work [BI89] on models with complex contour weights. See also ref. [Zah88] for earlier work on systems with complex interactions. The work presented here represents, to some extent, a combination, simplification and extension of much of the above work on Dobrushin states. In particular, we derive expansions for both the surface tension and expectation values in classical models with complex contour weights in which the phases are not related by symmetry. These features also occur in the quantum models we treat in ref. [BCF96].

The non-positivity of the weights leads to marked technical differences from the standard treatment. In theories with positive weights, control of the expansion for the surface tension automatically implies control of the probability distribution of the interfaces and hence control of the expectation values. Explicitly, standard Pirogov–Sinai theory of interfaces with positive weights (see e.g. ref. [HKZ88]) provides a representation of expectation values as sums over surfaces of conditional expectation values times the probabilities of these surfaces. In that case, one can obtain bounds on the expectation values by first deriving bounds on the conditional expectations and then resumming the probabilities. Here, in the absence of a probabilistic interpretation, we require some new techniques to control expectation values, in particular to establish exponential clustering in directions orthogonal to the interface. To this end, we will combine the cluster expansion for the complex surface weights and the expansion for the conditional expectations into a full-fledged cluster expansion for the non-translation-invariant state. See Section 6 for details.

While our methods and results apply to much more general models, for simplicity of exposition we formulate them here for a relatively simple model which shares two relevant features with more general models: namely, the contour weights are not necessary positive and there is no symmetry relating the two phases separated by the interface. The model we consider is a classical spin system on the lattice $\mathbb{Z}^d_{1/2} = \mathbb{Z}^d + (\frac{1}{2}, \frac{1}{2},..., \frac{1}{2})$ with spin configurations $s: \mathbb{Z}^d_{1/2} \to \{-1, +1\} : x \mapsto s_x$, and formal Hamiltonian

$$H = \sum_x H_x(s) \tag{1.1}$$

where $H_x(s)$ is a general complex-valued translation-invariant function of s. Here, we assume that the main contribution to $H_x(s)$ comes from an Ising interaction, but allow for small corrections whose main task is to break the $+/-$ symmetry of the Ising Hamiltonian. For concreteness, we set

$$H_x(s) = \frac{\beta}{4} \sum_{y:\, |x-y|=1} |s_x - s_y| - hs_x + \sum_{\substack{A:\, x \in A,\\ \operatorname{diam} A \leqslant 1}} \frac{\lambda_A}{|A|} \prod_{y \in A} s_y \tag{1.2}$$

where the first sum is over the $2d$ nearest neighbors of x,

$$\operatorname{diam} A := \sum_{x,\, y \in A} \max_{\mu = 1,..., d} |x_\mu - y_\mu| \tag{1.3}$$

β is the inverse temperature, h is the magnetic field, and λ_A are complex-valued coupling constants. Note that with this definition, we have absorbed the inverse temperature into the Hamiltonian. Furthermore, we will assume that the norm

$$\|\operatorname{Re} \lambda\| = \sum_{A:\, x \in A} |\operatorname{Re} \lambda_A| \tag{1.4}$$

is small compared to the real part of β, which will imply that the corresponding contour model has small activities for low temperatures.

Before describing our new results on non-translation-invariant states, let us briefly review what is known about *translation-invariant* Gibbs states of the model with Hamiltonian (1.1). Under the condition (1.4), one can apply the methods of Pirogov–Sinai theory in the form derived by Borgs and Imbrie [BI89]. Defining suitable metastable free energies $f_q = f_q(\beta, \lambda, h)$, $q = \pm 1$, one obtains a convergent cluster expansion for the state with constant boundary condition q whenever $\operatorname{Re} f_q = \min\{\operatorname{Re} f_+, \operatorname{Re} f_-\}$; in this case, one says that the boundary condition q is *stable*. In particular, on the hypersurface where $\operatorname{Re} f_+ = \operatorname{Re} f_-$, the cluster expansions for both the state with $+$ and the state with $-$ boundary conditions converge, and both boundary conditions are said to be stable. Whenever the boundary condition q is stable, one obtains existence of the infinite-volume limit, together with translation invariance and exponential clustering for this limit. For periodic boundary conditions, one also obtains the existence of the thermodynamic limit and a proof that this limit is an *equal weight* convex combination of the states that are stable at the given values of β, λ and h. Note that one finds equal weights even if there is no symmetry relating the two phases. While many models with real interactions allow for a proof of the fact that *all* translation invariant states are convex combinations of the corresponding stable states, such a statement is not proven for complex interactions.

The goal of this paper is the construction of infinite-volume states which break the translation invariance of the Hamiltonian. In particular, assuming $\operatorname{Re} f_+ = \operatorname{Re} f_-$, we show the existence of the thermodynamic limit of states with so-called Dobrushin boundary conditions, i.e. boundary conditions which introduce an interface parallel to one of the lattice hyperplanes. The resulting infinite-volume states exhibit exponential clustering, are translation invariant in directions parallel to the interface and break translation invariance in the direction orthogonal to the interface. Furthermore, these non-translation invariant infinite-volume states approach the corresponding translation-invariant pure states at distances asymptotically far from the interface; the rate of approach is exponential in the distance from the interface.

The organization of this paper is as follows. In Section 2, we define our notation and state our results. In Sections 3 and 4, we develop the expansions for the partition function and the states in a system with uniform boundary conditions, modifying the standard treatment where necessary. These results are then used in Sections 5 and 6 to develop expansions for the surface tension and the states in systems with Dobrushin boundary conditions.

2. DEFINITIONS AND STATEMENT OF RESULTS

Before stating our results, we must define the model in a finite volume and specify our boundary conditions. Given a finite box

$$\Lambda = \{x \in \mathbb{Z}^d_{1/2} \mid -L_i \leqslant x_i \leqslant L_i, i = 1,..., d\} \tag{2.1}$$

let $\Lambda^c = \mathbb{Z}^d_{1/2} \backslash \Lambda$ and let s_Λ be the collection of spin variables $\{s_x\}_{x \in \Lambda}$. We replace the Hamiltonian (1.1) by the finite-volume Hamiltonian[5]

$$H_\Lambda(s_\Lambda \mid s_{\Lambda^c}) = \sum_{x:\, \mathrm{dist}(x, \Lambda) \leqslant 1} H_x(s) \tag{2.2}$$

where $\mathrm{dist}(x, \Lambda)$ is the ℓ_∞-distance between x and Λ. One then defines the partition function $Z_\sigma(\Lambda)$ and finite-volume state $\langle \cdot \rangle_\sigma^\Lambda$ with boundary condition σ as

$$Z_\sigma(\Lambda) = \sum_{s_\Lambda} e^{-H_\Lambda(s_\Lambda \mid \sigma_{\Lambda^c})} \tag{2.3}$$

and

$$\langle A \rangle_\sigma^\Lambda = \frac{1}{Z_\sigma(\Lambda)} A(s)\, e^{-H_\Lambda(s_\Lambda \mid \sigma_{\Lambda^c})} \tag{2.4}$$

where $A = A(s)$ is a function depending only on the spins in Λ. Of particular interest for us are constant boundary conditions, $\sigma_x \equiv +$ or $\sigma_x \equiv -$, and Dobrushin boundary conditions

$$\sigma_x^{(+-)} = \begin{cases} + & \text{if} \quad x_1 > 0 \\ - & \text{if} \quad x_1 < 0 \end{cases} \tag{2.5}$$

where x_1 is the first component of $x \in \mathbb{Z}^d_{1/2}$. The corresponding partition functions will be denoted $Z_+(\Lambda), Z_-(\Lambda)$ and $Z_{+-}(\Lambda)$, and the corresponding states by $\langle \cdot \rangle_+^\Lambda$, $\langle \cdot \rangle_-^\Lambda$ and $\langle \cdot \rangle_{+-}^\Lambda$. Note that the choice of a box Λ invariant under reflection in the plane $x_1 = 0$, and the exact form of the boundary conditions (2.5), are not essential to the analysis in this paper, although different choices would substantially complicate the notation in Section 5 and 6.

Before stating our main result, we recall the known result for translation-invariant states $\langle \cdot \rangle_\pm$. In the following, we use the symbol $t_x(A)$ for the translation of a local observable A by a vector $x \in \mathbb{Z}^d$, where, as usual, a local observable is a bounded function $A(s)$ that depends only on a finite number of spin variables s_x.

Theorem 2.1 [BI89]. Let $d \geqslant 2$, and let $e_\pm(h)$ be the "ground state energies per site" $e_\pm(h) = H_x(s \equiv \pm 1)$. Then there exist constants

[5] Note that the condition $\mathrm{dist}(x, \Lambda) \leqslant 1$ in (2.2) stems from the constraint diam $A \leqslant 1$ in the expression (1.2) for H_x. If we instead used diam $A \leqslant R$ in (1.2), we would have $\mathrm{dist}(x, \Lambda) \leqslant R$ in (2.2).

$\tau_0 < \infty$, $\varepsilon > 0$ and $c < \infty$, and C^1 functions $f_\pm(h)$, "metastable free energies", such that the following statements hold provided

$$\tau \equiv \operatorname{Re} \beta - c \, \|\operatorname{Re} \lambda\| \geqslant \tau_0 \tag{2.6}$$

(i)

$$|f_\pm(h) - e_\pm(h)| + \left| \frac{d}{dh} (f_\pm(h) - e_\pm(h)) \right| \leqslant e^{-(\tau - \tau_0)} \tag{2.7}$$

(ii) If $\operatorname{Re} f_m(h) = f_0(h) \equiv \min\{\operatorname{Re} f_+(h), \operatorname{Re} f_-(h)\}$, then the

$$\langle A \rangle_m = \lim_{\Lambda \to \mathbb{Z}^d} \langle A \rangle_m^\Lambda \tag{2.8}$$

exists for all local observables A, is translation invariant, and

$$|\langle s_x \rangle_m - m| \leqslant e^{-(\tau - \tau_0)} \tag{2.9}$$

(iii) For all local observables A and B, there exists a constant $C_{AB} < \infty$ such that

$$|\langle A t_x(B) \rangle_m - \langle A \rangle_m \langle B \rangle_m| \leqslant C_{AB} e^{-\varepsilon(\tau - \tau_0)|x|} \tag{2.10}$$

provided $\operatorname{Re} f_m(h) = f_0(h)$.

(iv) Let $\langle \cdot \rangle_{\text{per}}^\Lambda$ be the finite-volume state with periodic boundary conditions. Then the limit

$$\langle A \rangle_{\text{per}} = \lim_{\Lambda \to \mathbb{Z}^d} \langle A \rangle_{\text{per}}^\Lambda \tag{2.11}$$

exists for all local observables A, and

$$\langle A \rangle_{\text{per}} = \frac{1}{|Q|} \sum_{m \in Q} \langle A \rangle_m \tag{2.12}$$

where $\Lambda = \{m \mid \operatorname{Re} f_m(h) = f_0(h)\}$. In particular,

$$\langle A \rangle_{\text{per}} = \tfrac{1}{2}(\langle A \rangle_+ + \langle A \rangle_-) \tag{2.13}$$

provided

$$\operatorname{Re} f_+(h) = \operatorname{Re} f_-(h) \tag{2.14}$$

The main result of this paper is the following analogue of Theorem 2.1 for nontranslation-invariant states.

Theorem 2.2. Let $\tau, \tau_0, \varepsilon$ and $f_{\pm}(h)$ be as in Theorem 2.1, and assume (2.14). Let

$$\Lambda(L_\perp, L) = \{x \in \mathbb{Z}^d_{1/2} \mid -L_\perp < x_1 < L_\perp,\ -L < x_i < L\ \forall i \geqslant 2\} \tag{2.15}$$

Then

(i) The limit

$$\langle A \rangle_{+-} = \lim_{L \to \infty} \lim_{L_\perp \to \infty} \langle A \rangle_{+-}^{\Lambda(L_\perp, L)} \tag{2.16}$$

exists for all local observables A and is translation invariant in the horizontal directions (i.e., in the directions orthogonal to the 1-direction).

(ii) For all local observables A, there exist constants $C_A < \infty$, such that

$$|\langle t_x(A) \rangle_{+-} - \langle A \rangle_+| \leqslant C_A e^{-\varepsilon(\tau - \tau_0)\,|x_1|} \tag{2.17}$$

if $x_1 > 0$, and

$$|\langle t_x(A) \rangle_{+-} - \langle A \rangle_-| \leqslant C_A e^{-\varepsilon(\tau - \tau_0)\,|x_1|} \tag{2.18}$$

if $x_1 < 0$.

(iii) For all local observables A and B there exist constants $C_{AB} < \infty$ such that

$$|\langle A t_x(B) \rangle_{+-} - \langle A \rangle_{+-} \langle t_x(B) \rangle_{+-}| \leqslant C_{AB} e^{-\varepsilon(\tau - \tau_0)\,|x|} \tag{2.19}$$

3. THE CLUSTER EXPANSION FOR $Z_{\pm}(\Lambda)$

In this section and the next, we review the cluster expansions for the translation-invariant partition functions $Z_{\pm}(\Lambda)$ and states $\langle \cdot \rangle_{\pm}$ (see ref. [BI89] and [BK90], Appendix A for details). Then in the following two sections, these expansions will be used to derive analogous quantities for the non-translation-invariant system. We begin by rewriting $Z_q(\Lambda)$ ($q = \pm 1$) as the partition function of a suitable contour model.

Given a box of the form (2.1), we let $\bar{\Lambda} = \{x \in \mathbb{Z}^d_{1/2} \mid \operatorname{dist}(x, \Lambda) \leqslant 1\}$, and let $V = V(\Lambda) \subset \mathbb{R}^d$ be the union of all closed unit cubes $c(x)$ with centers $x \in \bar{\Lambda}$. Given a configuration $s_\Lambda \in \{-1, 1\}^\Lambda$ and the boundary condition q, we extend s_Λ to $\bar{\Lambda}$ by setting $s_x = q$ if $x \in \bar{\Lambda} \backslash \Lambda$; we denote the resulting configuration by $s_{\bar{\Lambda}} = s_{\bar{\Lambda}}(s_\Lambda, q)$. We define $\bar{V}_m = \bar{V}_m(s_{\bar{\Lambda}}) \subset V$ as the union of all cubes $c(x) \subset V$ for which $s_x = m$. Finally, the set $\Gamma = \Gamma(s_{\bar{\Lambda}})$ is defined as $\bar{V}_+ \cap \bar{V}_-$, and the "ground state regions" $V_{\pm} = V_{\pm}(s_{\bar{\Lambda}})$ are defined as

$\bar{V}_{\pm}\backslash\Gamma$. In order to specify the configuration $s_{\bar{\Lambda}}$ uniquely, one has to decide which components of $V\backslash\Gamma$ are part of V_+ and which are part of V_-. To this end, one introduces contours with labels.

Given a configuration $s_{\bar{\Lambda}}$, the contours corresponding to $s_{\bar{\Lambda}}$ are defined as pairs $Y = (\text{supp } Y, \alpha)$, where $\text{supp } Y \subset \mathbb{R}^d$ is a connected component[6] of Γ, and the function α is an assignment of a label $\alpha(c) \in \{-1, +1\}$ to each cube that touches supp Y. (As usual, we say that two subsets of $\mathbb{R}^d$ touch if their intersection is nonempty.) The function α is constructed so that $\alpha(c) = m$ if $c \subset \bar{V}_m$. Note that the labels of contours corresponding to a configuration $s_{\bar{\Lambda}}$ are matching in the sense that the labels $\alpha(c)$ are constant on each component of $V\backslash\Gamma$.

On the other hand, a given set of contours $\{Y_1,..., Y_n\}$ corresponds to some configuration $s_{\bar{\Lambda}}$ if and only if

(i) $\text{supp } Y_i \subset V\backslash\partial V$ for all i,

(ii) $\text{supp } Y_i \cap \text{supp } Y_j = \varnothing$ for $i \neq j$, and

(iii) the labels of $Y_1,..., Y_n$ are matching.

We call a set of contours obeying (i) and (ii) a *set of non-overlapping contours in* V and a set of contours obeying (i)–(iii) a *set of non-overlapping contours in* V *with matching labels*, or sometimes just a *set of matching contours in* V.

In order to rewrite $Z_{\pm}(\Lambda)$ in terms of contours, we assign a weight $\rho(Y)$ to each contour $Y = (\text{supp } Y, \alpha)$. To this end, let s_Y be the configuration that is constant on each component of $\mathbb{R}^d\backslash\text{supp } Y$ and equal to $\alpha(c(x))$ whenever $c(x)$ is a cube that touches supp Y. Denoting the constant configurations on $\mathbb{Z}^d_{1/2}$ by $g^{(\pm)}$, $g^{(m)}_x \equiv m$, we then define

$$\rho(Y) = \exp\left(-\sum_{\substack{x \in \bar{\Lambda}: \\ c(x) \cap \text{supp } Y \neq \varnothing}} (H_x(s_Y) - e_{\alpha(x)})\right) \tag{3.1}$$

where

$$e_m = H_x(g^{(m)}) \tag{3.2}$$

is the energy per site in the ground state $g^{(m)}$. With this definition, the weight of a configuration $s_{\bar{\Lambda}} = s_{\bar{\Lambda}}(s_\Lambda, q)$ with contours $Y_1,..., Y_n$ is equal to

$$e^{-H(s_\Lambda \mid q)} = e^{-e_+ |V_+|} e^{-e_- |V_-|} \prod_{k=1}^{n} \rho(Y_k) \tag{3.3}$$

[6] Note that the edges are not rounded here, which implies that $\mathbb{R}^d\backslash\text{supp } Y$ may have more than one finite component.

implying that

$$Z_q(\Lambda) = \sum_{\{Y_1,\dots,Y_n\}} e^{-e_+ |V_+|} e^{-e_- |V_-|} \prod_{k=1}^{n} \rho(Y_k) \tag{3.4}$$

where the sum goes over all sets of matching contours in $V = V(\Lambda)$ with external boundary condition q. Here, as in the sequel, the external boundary condition of a set of contours $\{Y_1,\dots, Y_n\}$ in V is defined as the label of the component of $V \setminus (\text{supp } Y_1 \cup \cdots \cup \text{supp } Y_n)$ that touches ∂V.

Remarks. (i) Rewriting $H_x(s_Y)$ and $e_\alpha(x) = H_x(g^{(\alpha(x))})$ in terms of the right hand side of (1.2), it is easy to see that $\rho(Y)$ does not depend on s_x as soon as supp $Y \cap c(x) = \varnothing$.

(ii) For the standard Ising model (corresponding to $\lambda_A \equiv 0$), the weight of a contour Y is simply $\rho(Y) = e^{-\beta |Y|}$, where $|Y|$ is the $(d-1)$-dimensional area of supp Y.

(iii) For the more general model (1.2), the third term in (1.2) introduces corrections yielding a weight of the form $\rho(Y) = e^{-\beta |Y| + O(\|\lambda\| \, |Y|)}$. As a consequence,

$$|\rho(Y)| \leqslant e^{-\tau |Y|} \tag{3.5}$$

where τ is given in Theorem 2.1.

(iv) With a slight abuse of notation, we will use the symbol $Z_q(V)$ for the partition function $Z_q(\Lambda)$ if $V = V(\Lambda)$, where Λ is a box of the form (2.1) and $V(\Lambda)$ is obtained by "fattening" Λ as described at the beginning of this section. We will also need more general volumes V which are unions of closed unit cubes, but which cannot be obtained by fattening any Λ of the form (2.1). If V is such a volume and has no holes (in the sense that $V^c = \mathbb{R}^d \setminus V$ is a *connected* subset of $\mathbb{R}^d$), we will use the symbol $Z_q(V)$ for the contour partition function defined by the right hand side of (3.4). Finally, if V is a union of closed unit cubes with holes, $Z_q(V)$ is defined by requiring that no contour in (3.4) surround the holes of V. From now on, this condition—as well as the condition that Y does not touch the boundary of V—will be implicit in statements like "Y is a contour in V," or "where the sum goes over sets of non-overlapping contours in V." Finally, we will often use the notation $Z_q(V)$ for the partition function of a volume V which is the interior of a contour, and which is therefore an open set. In this case, $Z_q(V)$ is defined as $Z_q(\bar{V})$ where $\bar{V} = V \cup \partial V$.

In order to apply the usual techniques of Mayer expansions for abstract polymer systems (see e.g. [Sei82], [GJ85] or [Bry86]), one needs a representation for $Z_{\pm}(V)$ that eliminates the matching condition on contours. We introduce some notation. For a contour Y with support supp Y, we define Int Y as the union of all finite components of $\mathbb{R}^d\backslash$supp Y, $V(Y) = \text{supp } Y \cup \text{Int } Y$, and Ext $Y = V\backslash V(Y)$. We use the symbol $\text{Int}_m Y$ to denote the union of all components of Int Y that carry the label m, $m = \pm 1$. We say that Y is an m contour if the label $\alpha(c) = m$ for all cubes c in Ext Y that touch the support of Y. Finally we say that Y is an external contour in the set $Y_1,..., Y_n$ if there is no contour Y_i, $i = 1,..., n$, such that supp $Y \subset \text{Int } Y_i$. Note that all external contours contributing to $Z_+(V)$ are $+$ contours, while all external contours contributing to $Z_-(V)$ are $-$ contours. We now resum (3.4) inside $\text{Int}_m Y$ for all external contours Y. This resummation produces a factor $Z_m(\text{Int}_m Y)$ for each external Y and each $m = \pm$, and yields the expression

$$Z_q(V) = \sum_{\{Y_1,..., Y_k\}_{\text{ext}}} e^{-|\text{Ext}|\, e_q} \prod_{i=1}^{k} \left(\rho(Y_i) \prod_{m=\pm 1} Z_m(\text{Int}_m Y_i) \right) \tag{3.6}$$

where the sum goes over sets $\{Y_1,..., Y_k\}_{\text{ext}}$ of mutually external q contours and $\text{Ext} = V\backslash(V(Y_1) \cup \cdots \cup V(Y_k))$. Next, we divide each $Z_m(\text{Int}_m Y_i)$ by $Z_q(\text{Int}_m Y_i)$ and multiply it back in the form (3.6), a standard device in Pirogov–Sinai theory (see e.g. [PS75] and [Zah84]). Iterating this process, one obtains the desired representation

$$Z_q(V) = e^{-e_q |V|} \sum_{\{Y_1,..., Y_n\}} \prod_{i=1}^{n} K_q(Y_i) \tag{3.7}$$

where the sum now goes over sets of non-overlapping q contours in V without any matching condition, and

$$K_q(Y) = \rho(Y) \prod_m \frac{Z_m(\text{Int}_m Y)}{Z_q(\text{Int}_m Y)} \tag{3.8}$$

Applied to the model (1.2), the key estimate of [BI89], see also [BK90], Appendix A, is the following:

Proposition 3.1 [BI89], [BK90]. Let τ, τ_0, $f_{\pm}(h)$ and $f_0(h)$ be as in Theorem 2.1. If Re $f_q(h) = f_0(h)$, then

$$|K_q(Y)| \leqslant e^{-(\tau - O(1))\,|Y|} \tag{3.9}$$

for all m contours Y, and

$$f_q(h) = -\lim_{V \to \mathbb{Z}^d} \frac{1}{|V|} \log Z_q(V) \tag{3.10}$$

where the limit is taken through any sequence of volumes such that $|\partial V|/|V| \to 0$.

Henceforth, we will assume that we are in the coexistence regime, i.e. the regime where the condition (2.14) is satisfied. Then we can use Proposition 3.1 to obtain convergent expansions for the partition functions $Z_+(V)$ and $Z_-(V)$. The usual Mayer expansion for abstract polymer systems then gives the representation

$$\log Z_q(V) = -e_\pm(h)\,|V| + \sum_{n=1}^{\infty} \sum_{Y_1,\dots,Y_n} \frac{\phi_c(Y_1,\dots,Y_n)}{n!} \prod_{k=1}^{n} K_q(Y_k) \tag{3.11}$$

where the second sum runs over sequences of m contours in V, and $\phi_c(Y_1,\dots,Y_n)$ is a combinatoric factor. In terms of the connectivity graph $G(Y_1,\dots,Y_n)$ on $1,\dots,n$, which has a line between i and j whenever $\operatorname{supp} Y_i \cap \operatorname{supp} Y_j \neq \varnothing$, ϕ_c is defined as

$$\phi_c(Y_1,\dots,Y_n) = \sum_{C \subset G(Y_1,\dots,Y_n)} (-1)^{\ell(C)} \tag{3.12}$$

where the sum goes over all connected subgraphs on $1,\dots,n$, and $\ell(C)$ is the number of lines in C. The factor $\phi_c(Y_1,\dots,Y_n)$ is zero if $G(Y_1,\dots,Y_n)$ is not a connected graph on $1,\dots,n$.

Fixing the set $X = \operatorname{supp} Y_1 \cup \cdots \cup \operatorname{supp} Y_n$ and resumming over all n and all sequences of contours that lead to the same set X, we finally get a representation for $\log Z_q$ in terms of connected sets $X \subset \mathbb{R}^d$, henceforth called *clusters*, which are unions of (the supports of) a finite number of contours:

$$\log Z_q(V) = -e_\pm(h)\,|V| + \sum_X k_q(X) \tag{3.13}$$

where the sum goes over all clusters X in V that do not touch the boundary ∂V of V, and

$$k_q(X) = \sum_{n=1}^{\infty} \sum_{\substack{Y_1,\dots,Y_n:\\ \bigcup_i \operatorname{supp} Y_i = X}} \frac{\phi_c(Y_1,\dots,Y_n)}{n!} \prod_{k=1}^{n} K_q(Y_k) \tag{3.14}$$

By the bound (3.9) and the well known properties of the Mayer expansion for abstract polymer systems,

$$|k_q(X)| \leqslant e^{-(\tau - O(1))\,|X|} \tag{3.15}$$

The representation (3.14) allows us to control the boundary effects. Consider a volume V without holes. Let $\bar{V} = V \cup \partial V$, and let V_0 be the union over all cubes in $\bar{V}$ that do not touch ∂V. Also, for a cluster X, let $\operatorname{Int} X$ be the union of all finite components of $\mathbb{R}^d \backslash X$, let $V(X) = X \cup \operatorname{Int} X$ and let $\bar{X}$ be the "fattened cluster" consisting of all cubes $c(x)$ in $V(X)$ that touch the set X. Making explicit the condition in the sum in (3.13), we require that $\bar{X} \subset V_0$. It then follows from (3.13) and (3.10) that the free energy of the phase q may be identified as

$$f_q(h) = e_q(h) - \sum_{X:\, c(0) \subset \bar{X}} \frac{1}{|\bar{X}|} k_q(X) \tag{3.16}$$

with the finite-volume correction

$$\begin{aligned} \log Z_q(V) + f_q(h)\,|V| &= - \sum_{c(x) \subset \bar{V}} \sum_{\substack{X:\, c(x) \subset \bar{X} \\ \bar{X} \not\subset V_0}} \frac{1}{|\bar{X}|} k_q(X) \\ &= - \sum_{X:\, \bar{X} \cap \partial V \neq \varnothing} \frac{|\bar{X} \cap \bar{V}|}{|\bar{X}|} k_q(X) \end{aligned} \tag{3.17}$$

Remarks. (i) For volumes of the form $V = \bar{V} = \{x \in \mathbb{R}^d \mid -L_i \leqslant x_i \leqslant L_i\}$, it is sometimes convenient to write the right hand side of (3.17) as a sum over clusters $X \subset V$. To this end, we introduce the projection $P_V\colon \mathbb{R}^d \to V$, where $P_V(x)$ has components

$$(P_V(x))_i = \begin{cases} x_i & \text{if } |x_i| \leqslant L_i \\ L_i & \text{if } x_i > L_i \\ -L_i & \text{if } x_i < -L_i \end{cases} \tag{3.18}$$

For $X \subset V$, we then define

$$k_{q,V}(X) = \sum_{X':\, X = P_V(X')} \frac{|\bar{X}|}{|\bar{X}'|} k_q(X') \tag{3.19}$$

leading to

$$\log Z_q(V) + f_q(h)\,|V| = - \sum_{\substack{X:\, X \subset V \\ \bar{X} \cap \partial V \neq \varnothing}} k_{q,V}(X) \tag{3.20}$$

(ii) Let V be a rectangular volume of the form considered in (i), and let $V_1 \subset V$ be an arbitrary subvolume. Then

$$\log Z_q(V_1) + f_q(h)\,|V_1| = -\sum_{\substack{X:\, X \subset V \\ \bar{X} \cap \partial V_1 \neq \varnothing}} \frac{|\bar{X} \cap \bar{V}_1|}{|\bar{X}|} k_{q,V}(X) \tag{3.21}$$

4. CLUSTER EXPANSION FOR THE EXPECTATION VALUES $\langle \cdot \rangle_\pm$

In order to derive a cluster expansion for $\langle \cdot \rangle_q$, $q = \pm$, we first consider the unnormalized expectation values

$$Z_q(A; V) = Z_q(V) \langle A \rangle_q^V \tag{4.1}$$

of local observables A. As before, the spin configuration $s_{\bar{A}}$ is uniquely defined by the corresponding contours $Y_1,..., Y_n$. Let $\mathscr{A} \subset \mathbb{Z}^d_{1/2}$ denote the set of sites on which the local observable A depends: $A(s) = A(\{s_x\}_{x \in \mathscr{A}})$, and let $\operatorname{supp} A = \bigcup_{x \in \mathscr{A}} c(x)$. Observing that only those contours which surround or intersect the set $\mathscr{A}$ can influence the value of A, we group all contours Y_i with $V(Y_i) \cap \operatorname{supp} A \neq 0$ into a new "contour" Y_A, and introduce the sets

$$\operatorname{supp} Y_A = \bigcup_{Y \in Y_A} \operatorname{supp} Y, \qquad V(Y_A) = \bigcap_{Y \in Y_A} V(Y)$$

$$\operatorname{Int} Y_A = V(Y_A) \backslash \operatorname{supp} Y_A \qquad \text{and} \qquad \operatorname{Ext} Y_A = V \backslash V(Y_A)$$

as well as

$$\operatorname{Int}^{(0)} Y_A = \operatorname{Int} Y_A \backslash \operatorname{supp} A \qquad \text{and} \qquad \operatorname{Ext}^{(0)} Y_A = \operatorname{Ext} Y_A \backslash \operatorname{supp} A$$

As usual, $\operatorname{Int}_m Y_A$ is the union of all components of $\operatorname{Int} Y_A$ which have boundary condition m, $\operatorname{Int}_m Y_A = \operatorname{Int} Y_A \cap V_m$, while $\operatorname{Int}_m^{(0)} Y_A = \operatorname{Int}^{(0)} Y_A \cap V_m$. Notice that Y_A may be the empty set, in which case $\operatorname{Int} Y_A = \varnothing$ and $\operatorname{Ext} Y_A = V$.

Recalling that A depends only on those contours for which $V(Y) \cap \operatorname{supp} A \neq \varnothing$, we define

$$\rho_A(Y_A) = A(Y'_1,..., Y'_{n'})\, e^{-e_+ |V_+ \cap \operatorname{supp} A|}\, e^{-e_- |V_- \cap \operatorname{supp} A|} \prod_{k=1}^{n'} \rho(Y'_k) \tag{4.2}$$

where $Y'_1,..., Y'_{n'}$ are the contours in Y_A. Fixing now, for a moment, all contours Y_i in (4.1) for which $V(Y_i) \cap \operatorname{supp} A \neq \emptyset$, and resumming the rest, we obtain

$$Z_q(A; V) = \sum_{Y_A} \rho_A(Y_A)\, Z_q(\operatorname{Ext}^{(0)} Y_A) \prod_{m=\pm 1} Z_m(\operatorname{Int}_m^{(0)} Y_A) \tag{4.3}$$

Introducing

$$K_{q,A}(Y_A) = \rho_A(Y_A)\, e^{e_q |\operatorname{supp} A|} \prod_{m=\pm 1} \frac{Z_m(\operatorname{Int}_m^{(0)} Y_A)}{Z_q(\operatorname{Int}_m^{(0)} Y_A)} \tag{4.4}$$

we further rewrite (4.3) as

$$Z_q(A; V) = \sum_{Y_A} K_{q,A}(Y_A)\, e^{-e_q |\operatorname{supp} A|} Z_q(\operatorname{Ext}^{(0)} Y_A)\, Z_q(\operatorname{Int}^{(0)} Y_A) \tag{4.5}$$

Using finally the representation (3.7) for $Z_q(\operatorname{Ext}^{(0)} Y_A)$ and $Z_q(\operatorname{Int}^{(0)} Y_A)$, we get

$$Z_q(A; V) = e^{-e_q |V|} \sum_{Y_A} K_{q,A}(Y_A) \sum_{\{Y_1,..., Y_n\}} \prod_{k=1}^{n} K_q(Y_k) \tag{4.6}$$

Here the second sum goes over sets of non-overlapping q contours $Y_1,..., Y_n$, such that for all contours Y_i, the set $V(Y_i)$ does not intersect the set $\operatorname{supp} Y_A \cup \operatorname{supp} A$.

In order to make the connection to the standard Mayer expansion for polymer systems, we now introduce $G(Y_A, Y_1,..., Y_n)$ as the graph on the vertex set $\{0, 1,..., n\}$ which has an edge between two vertices $i \geqslant 1$ and $j \geqslant 1$, $i \neq j$, whenever $\operatorname{supp} Y_i \cap \operatorname{supp} Y_j \neq \emptyset$, and an edge between the vertex 0 and a vertex $i \neq 0$ whenever $V(Y_i) \cap (\operatorname{supp} Y_A \cup \operatorname{supp} A) \neq 0$. Implementing the non-overlap constraint in (4.6) by a characteristic function $\phi(Y_A, Y_1,..., Y_n)$ which is zero whenever the graph G has less than $n+1$ components, the standard Mayer expansion for polymer systems (see, for example, [Sei82]) then yields

$$\langle A \rangle_q^V = \frac{Z_q(A; V)}{Z_q(V)} = \sum_{n=0}^{\infty} \sum_{Y_A} \sum_{Y_1,..., Y_n} \frac{\phi_c(Y_A, Y_1,..., Y_n)}{n!} K_{q,A}(Y_A) \left[\prod_{k=1}^{n} K_q(Y_k) \right] \tag{4.7}$$

In terms of the graph $G(Y_A, Y_1,..., Y_n)$, the combinatoric factor $\phi_c(Y_A, Y_1,..., Y_n)$ is again defined by (3.12). It vanishes if the graph $G(Y_A, Y_1,..., Y_n)$ has more than one component.

As before, it is convenient to resum this expansion in terms of clusters. We define an *A-cluster* as a set $X_A \subset \mathbb{R}^d$ that is a union of finitely many clusters and has connected components X with $V(X) \cap \operatorname{supp} A \neq 0$, and its weight $k_{q,A}(X_A)$ as

$$k_{q,A}(X_A) = \sum_{n=0}^{\infty} \sum_{Y_A, Y_1,..., Y_n}' \frac{\phi_c(Y_A, Y_1,..., Y_n)}{n!} K_{q,A}(Y_A) \left[\prod_{k=1}^{n} K_q(Y_k) \right] \tag{4.8}$$

where the sum $\sum'$ goes over all sequences Y_A, Y_1,..., Y_n with $\operatorname{supp}[Y]_A \cup \operatorname{supp} Y_1 \cup \cdots \cup \operatorname{supp} Y_n = X_A$. With these definitions, we get

$$\langle A \rangle_q^V = \sum_{X_A} k_{q,A}(X_A) \tag{4.9}$$

where the sum goes over all A-clusters X_A in V that do not touch the boundary ∂V of V.

Remarks. (i) Note that our definition of A-clusters implies that X_A may be the empty set. In this case, $k_{q,A}(Y_A) = A(g^{(q)})$, where $g^{(q)}$ is the ground state introduced before equation (3.1).

(ii) Using the representation (3.17) for $Z_m(\operatorname{Int}_m^{(0)} Y_A)$ and $Z_q(\operatorname{Int}_m^{(0)} Y_A)$ and the fact that $\operatorname{Re} f_+ = \operatorname{Re} f_-$ in the coexistence regime to estimate the ratio of partition functions in (4.4), and using the bound (2.7) together with the coexistence condition to estimate $|\operatorname{Re} e_+ - \operatorname{Re} e_-|$, one gets the bound

$$|K_{q,A}(Y_A)| \leqslant \|A\| \, e^{O(e^{-\tau}) |\operatorname{supp} A|} e^{-(\tau - O(1)) |Y_A|} \tag{4.10}$$

where $\|A\|$ is the L_∞ norm of A and

$$|Y_A| = \sum_{Y \in Y_A} |\operatorname{supp} Y| \tag{4.11}$$

As a consequence, $k_{q,A}(Y_A)$ obeys a bound of the same form,

$$|K_{q,A}(Y_A)| \leqslant \|A\| \, e^{O(e^{-\tau}) |\operatorname{supp} A|} e^{-(\tau - O(1)) |Y_A|} \tag{4.12}$$

and

$$|\langle A \rangle_q^V| \leqslant \|A\| \, e^{O(e^{-\tau}) |\operatorname{supp} A|} \tag{4.13}$$

uniformly in V.

5. INTERFACES, WALLS, AND SURFACE TENSION

In this section, we analyze the partition function Z_{+-}, giving, in particular, a convergent low-temperature expansion for the surface tension

$$\sigma = -\lim_{L\to\infty} \lim_{L_\perp\to\infty} \frac{1}{L^{d-1}} \log \frac{Z_{+-}(\Lambda(L_\perp, L))}{\sqrt{Z_+(\Lambda(L_\perp, L))\, Z_-(\Lambda(L_\perp, L))}} \tag{5.1}$$

where $\Lambda(L_\perp, L)$ is defined in (2.15). For the standard Ising model with real $\beta \gg 1$, such an expansion was first derived in [Gal72] and [Dob72]. The methods presented here are an adaption of the expansions developed in [BF85] and [HKZ88] (see also [BI92], Section 5) to models with complex contour activities

Throughout this and the following section, $\Lambda, \Lambda_\infty, V$ and V_∞ will denote the volumes

$$\Lambda = \Lambda(L_\perp, L) = \{x \in \mathbb{Z}^d_{1/2} \mid |x_1| < L_\perp, |x_i| < L \ \forall i \geq 2\} \tag{5.2a}$$

$$\Lambda_\infty = \Lambda_\infty(L) = \{x \in \mathbb{Z}^d_{1/2} \mid |x_i| < L \ \forall i \geq 2\} \tag{5.2b}$$

$$V = V(L_\perp, L) = \{x \in \mathbb{R}^d \mid |x_1| < L_\perp + 1, |x_i| < L + 1 \ \forall i \geq 2\} \tag{5.2c}$$

$$V_\infty = V_\infty(L) = \{x \in \mathbb{R}^d \mid |x_i| < L + 1 \ \forall i \geq 2\} \tag{5.2d}$$

Consider a configuration $s_{\bar\Lambda}$ contributing to $Z_{+-}(\Lambda)$ and the corresponding set Γ separating the regions where $s_x = +1$ from the regions where $s_x = -1$. The set Γ now consists of one component $\operatorname{supp} S$ whose boundary lies in ∂V: $\partial \operatorname{supp} S = \{x \in \partial V : x_1 = 0\}$, and a finite number of components $\operatorname{supp} Y_1, ..., \operatorname{supp} Y_n$ which do not touch ∂V. As before, we define the contours $Y_1, ..., Y_n$ corresponding to the configuration $s_{\bar\Lambda}$ as the pairs $Y_i = (\operatorname{supp} Y_i, \alpha_i(\cdot))$, where α_i is an assignment of a label $\alpha_i(c(x)) = s_x$ to each cube $c(x)$ that touches $\operatorname{supp} Y_i$. In a similar way, the *interface* S is defined as the pair $S = (\operatorname{supp} S, \alpha_S(\cdot))$, where again α_S is the assignment of a label $\alpha_S(c(x)) = s_x$ to each cube $c(x)$ touching $\operatorname{supp} S$.

Defining $\rho(S)$ in the same way as $\rho(Y)$, see Eq. (3.1), we get the analogue of Eq. (3.3):

$$e^{-H(s_\Lambda \mid +-)} = e^{-e_- |V_-| - e_+ |V_+|} \rho(S) \prod_{i=1}^{n} \rho(Y_i) \tag{5.3}$$

where $e_\pm$ are given by (3.2) and $V_\pm = V_\pm(S, Y_1, ..., Y_n)$ are the regions where $s_x = \pm 1$. Furthermore

$$\rho(S) \leq e^{-\tau |\operatorname{supp} S|} \tag{5.4}$$

As a consequence of (5.3), we have the analogue of (3.4):

$$Z_{+-}(\Lambda) = \sum_{\{S, Y_1,..., Y_n\}} e^{-e_- |V_-| - e_+ |V_+|} \rho(S) \prod_{i=1}^{n} \rho(Y_i) \tag{5.5}$$

As before, we will identify $Z_{+-}(V)$ as $Z_{+-}(\Lambda)$ if $V = V(\Lambda)$. Resumming the contours $Y_1,..., Y_n$, this leads to the expression

$$Z_{+-}(V) = \sum_{S} \rho(S) Z_+(V_+(S)) Z_-(V_-(S)) \tag{5.6}$$

where $V_q(S)$ is defined as the union of all components of $V \backslash \operatorname{supp} S$ that carry the label q, $q = \pm 1$. Note that $V_q(S)$ consists of one component that touches the boundary of V (we denote this component by $\operatorname{Ext}_q S$), and possibly several components that are entirely bounded by $\operatorname{supp} S$ (we denote the union of these components by $\operatorname{Int}_q S$).

Extracting a factor $\rho(S_o) \sqrt{Z_+(V) Z_-(V)}$ from $Z_{+-}(V)$, where S_0 is the minimal interface compatible with the boundary conditions $\sigma^{(+-)}$,

$$\operatorname{supp} S_0 = \{x \in \mathbb{R}^d \mid x_1 = 0, |x_i| \leqslant L + 1 \ \forall i \geqslant 2\}$$

we rewrite (5.6) as

$$Z_{+-}(V) = \rho(S_o) \sqrt{Z_+(V) Z_-(V)}\, \tilde{Z}_{+-}(V) \tag{5.7}$$

where

$$\tilde{Z}_{+-}(V) = \sum_{S} \frac{Z_+(V_+(S)) Z_-(V_-(S))}{\sqrt{Z_+(V) Z_-(V)}} \frac{\rho(S)}{\rho(S_o)} \tag{5.8}$$

Equation (5.8) gives $\tilde{Z}_{+-}(V)$ as a sum over interfaces with *a priori* weight $\rho(S)/\rho(S_0)$ and an interaction in terms of a ratio of partition functions. In order to analyze this interaction, we will use the expansions (3.20) and (3.21) to write the interaction in terms of clusters which intersect the interface S. We introduce the sets $\partial_\pm V = \{x \in \partial V \mid \pm x_1 > 0\}$, and rewrite the expansion (3.20) for $\log Z_q(V)$ as

$$\begin{aligned}
\log Z_q(V) &= -f_q(h)\,|V| - \sum_{\substack{X:\, X \subset V \\ \bar{X} \cap \partial V \neq \varnothing}} k_{q,V}(X) \\
&= -2 f_q(h)\,|V_q(S_0)| - \sum_{\substack{X:\, \bar{X} \cap \partial_+ V \neq \varnothing \\ \bar{X} \cap \partial_- V \neq \varnothing}} k_{q,V}(X) \\
&\quad - \sum_{\substack{X:\, \bar{X} \cap \partial_+ V = \varnothing \\ \bar{X} \cap \partial_- V \neq \varnothing}} k_{q,V}(X) - \sum_{\substack{X:\, \bar{X} \cap \partial_+ V \neq \varnothing \\ \bar{X} \cap \partial_- V = \varnothing}} k_{q,V}(X)
\end{aligned} \tag{5.9}$$

where we have used that $|V_\pm(S_0)| = |\{x \in V \mid \pm x_1 > 0\}| = \frac{1}{2}|V|$. Using the symmetry under reflections at the $x_1 = 0$ plane, we therefore get

$$\frac{1}{2}\log Z_q(V) = -f_q(h)\,|V_q(S_0)| - \frac{1}{2}\sum_{\substack{X:\,\bar{X}\cap\partial_+ V\neq\varnothing\\ \bar{X}\cap\partial_- V\neq\varnothing}} k_{q,V}(X) - \sum_{\substack{X:\,\bar{X}\cap\partial_q V\neq\varnothing\\ \bar{X}\cap\partial_{-q} V=\varnothing}} k_{q,V}(X) \tag{5.10}$$

where $-q = +1$ if $q = -1$ and vice versa.

For $\log Z_q(V_q(S))$, we use the fact that $\partial V_q(S) = \operatorname{supp} S \cup \partial_q V$ to rewrite the expansion (3.21) as

$$\begin{aligned}\log Z_q(V_q(S)) &= -f_q(h)\,|V_q(S)| - \sum_{X\subset V:\,\bar{X}\cap\partial V_q(S)\neq\varnothing} \frac{|\bar{X}\cap\bar{V}_q(S)|}{|\bar{X}|} k_{k,V}(X)\\ &= -f_q(h)\,|V_q(S)| - \sum_{X\subset V:\,\bar{X}\cap\operatorname{supp} S\neq\varnothing} \frac{|\bar{X}\cap\bar{V}_q(S)|}{|\bar{X}|} k_{q,V}(X)\\ &\quad - \sum_{\substack{X\subset V:\,\bar{X}\cap\operatorname{supp} S=\varnothing\\ \bar{X}\cap\partial_q V\neq\varnothing}} \frac{|\bar{X}\cap\bar{V}_q(S)|}{|\bar{X}|} k_{q,V}(X)\end{aligned} \tag{5.11}$$

For $X \subset V$, the two conditions $\bar{X}\cap\operatorname{supp} S = \varnothing$ and $\bar{X}\cap\partial_q V \neq \varnothing$ imply that $\bar{X}\cap\partial_{-q} V = \varnothing$ and $\bar{X}\cap\bar{V}_q(S) = \bar{X}\cap V = \bar{X}$. Thus

$$\begin{aligned}\log Z_q(V_q(S)) &= -f_q(h)\,|V_q(S)| - \sum_{X\subset V:\,\bar{X}\cap\operatorname{supp} S\neq\varnothing} \frac{|\bar{X}\cap\bar{V}_q(S)|}{|\bar{X}|} k_{q,V}(X)\\ &\quad - \sum_{\substack{X\subset V:\,\bar{X}\cap\operatorname{supp} S=\varnothing\\ \bar{X}\cap\partial_q V\neq\varnothing\\ \bar{X}\cap\partial_{-q} V=\varnothing}} k_{q,V}(X)\\ &= -f_q(h)\,|V_q(S)| - \sum_{\substack{X\subset V:\,\bar{X}\cap\partial_q V\neq\varnothing\\ \bar{X}\cap\partial_{-q} V=\varnothing}} k_{q,V}(X)\\ &\quad + \sum_{\substack{X\subset V:\,\bar{X}\cap\operatorname{supp} S\neq\varnothing\\ \bar{X}\cap\partial_q V\neq\varnothing\\ \bar{X}\cap\partial_{-q} V\neq\varnothing}} k_{q,V}(X)\\ &\quad - \sum_{X\subset V:\,\bar{X}\cap\operatorname{supp} S\neq\varnothing} \frac{|\bar{X}\cap\bar{V}_q(S)|}{|\bar{X}|} k_{q,V}(X)\end{aligned} \tag{5.12}$$

where we have used inclusion-exclusion in the second step. Combining (5.8) with (5.10) and (5.12), we get the representation

$$\tilde{Z}_{+-}(V) = \sum_S \frac{\rho(S)}{\rho(S_o)} e^{\Delta F(S) + W_V(S)} \tag{5.13}$$

where

$$\Delta F(S) = f_+(h)\,\Delta_+(S) + f_-(h)\,\Delta_-(h) \tag{5.14}$$

$$\Delta_q(S) = |V_q(S_0)| - |V_q(S)| \tag{5.15}$$

while $W_V(S)$ is a sum over clusters X that intersect S,

$$W_V(S) = \sum_{q=\pm 1} \Bigg(\sum_{\substack{X \subset V:\, \bar{X} \cap \operatorname{supp} S \neq \varnothing \\ \bar{X} \cap \partial_q V \neq \varnothing \\ \bar{X} \cap \partial_{-q} V \neq \varnothing}} k_{q,V}(X) + \frac{1}{2} \sum_{\substack{X:\, \bar{X} \cap \partial_+ V \neq \varnothing \\ \bar{X} \cap \partial_- V \neq \varnothing}} k_{q,V}(X) - \sum_{X \subset V:\, \bar{X} \cap \operatorname{supp} S \neq \varnothing} \frac{|\bar{X} \cap V_q(S)|}{|\bar{X}|} k_{q,V}(X) \Bigg) \tag{5.16}$$

Introducing the notation $X \leftrightarrow S$ for a cluster X with $\bar{X} \cap \operatorname{supp} S \neq \varnothing$, we get a representation of the form

$$W_V(S) = \sum_{X \leftrightarrow S} k_V(X, S) \tag{5.17}$$

where $k_V(X, S)$ is an activity which, by (3.15), decays exponentially with the size of X,

$$|k_V(X, S)| \leqslant e^{-(\tau - O(1))\,|X|} \tag{5.18}$$

Lemma 5.1. Let $d \geqslant 2$ and $\operatorname{Re} f_+(h) = \operatorname{Re} f_-(h)$. Then there is a constant $\tau_o < \infty$ such that the limit

$$\tilde{Z}_{+-}(V_\infty) = \lim_{L_\perp \to \infty} \tilde{Z}_{+-}(V(L_\perp, L)) \tag{5.19}$$

exists and is equal to

$$\tilde{Z}_{+-}(V_\infty) = \sum_S \frac{\rho(S)}{\rho(S_o)} e^{\Delta F(S) + W(S)} \tag{5.20}$$

provided $\tau \geqslant \tau_0$. The sum in (5.20) goes over all finite interfaces in V_∞, and

$$W(S) = \sum_{X \leftrightarrow S} k(X, S) \tag{5.21}$$

where

$$k(X, S) = \lim_{L_\perp \to \infty} k_{V(L_\perp, L)}(X, S) \tag{5.22}$$

Proof. Let us first observe that the number of connected clusters $X \leftrightarrow S$ that have size $|X| = s$ is bounded by $|\text{supp}\, S|$ times a geometrical constant to the power s. Thus, by (5.18), the sum in (5.17) is absolutely convergent, giving the bound $W_V(S) \leqslant O(e^{-\tau})\, |\text{supp}\, S|$. Since, on the other hand,

$$\Delta_+(S) + \Delta_-(S) = 0 \tag{5.23}$$

and $\text{Re}\, f_+(h) = \text{Re}\, f_-(h)$, we have

$$|e^{\Delta F(S) + W_V(S)}| = |e^{W_V(S)}| \leqslant e^{O(e^{-\tau})\, |\text{supp}\, S|} \tag{5.24}$$

uniformly in $L_\perp$. Inserting (5.94) into (5.13), and using (5.4), this gives absolute convergence uniformly in $L_\perp$. By dominated convergence, it is therefore enough to show convergence term by term, i.e. convergence of $W_V(S)$ to $W(S)$. This, in turn, immediately follows from the uniform convergence of the cluster expansion (5.17). ∎

Given the representation (5.20), we now expand $e^{W(S)}$ about 1. To this end, we define the set

$$\mathbb{X}(S) = \{X \mid X \leftrightarrow S\} \tag{5.25}$$

and rewrite $e^{W(S)}$ as

$$e^{W(S)} = \prod_{X \leftrightarrow S} e^{k(X, S)} = \sum_{\mathbb{X} \subset \mathbb{X}(S)} \prod_{X \in \mathbb{X}} (e^{k(X, S)} - 1) \tag{5.26}$$

For each $\mathbb{X}$, we then decompose the set $\bigcup \{Y \mid Y \in \mathbb{X}\}$ into its connected components $X_1, \ldots, X_m$, leading to the expansion

$$e^{W(S)} = \sum_{\{X_1, \ldots, X_m\}} \prod_i z(X_i, S) \tag{5.27}$$

where

$$z(X, S) := \sum_{\mathbb{X}:\, X = \cup\{Y \mid Y \in \mathbb{X}\}} (e^{k(X, S)} - 1) \tag{5.28}$$

which, by (5.18), decays exponentially in the size of X:

$$|z(X, S)| \leqslant e^{-(\tau - O(1))\,|X|} \tag{5.29}$$

Inserting (5.27) into (5.20), we get

$$\tilde{Z}_{+-}(V_\infty) = \sum_S \sum_{\{X_1,..., X_m\}} \frac{\rho(S)}{\rho(S_o)} e^{\Delta F(S)} \prod_{j=1}^{m} z(X_j, S) \tag{5.30}$$

where the second sum goes over all sets of clusters $X_1,..., X_m$ such that $X_i \leftrightarrow S$ for all i and $X_i \not\leftrightarrow X_j$ for all $i \neq j$.

In order to continue, we need some notation. We refer to the directions parallel to S_0 as horizontal, and to the direction orthogonal to S_0 as vertical. We define π as the orthogonal projection from V_∞ onto supp S_0, $\pi(x) = (0, x_2,..., x_d)$, and $h(x)$ as the height of a point $x \in \mathbb{R}^d$, $h(x) = x_1$. Finally, we say that p is a plaquette in an interface S if $p \subset \operatorname{supp} S$ is the face of a unit cube $c(x)$ with center $x \in \mathbb{Z}^d_{1/2}$.

Given an interface S and a set of clusters $\{X_1,..., X_m\}$ connected to S, define the decorated interface S^{dec} as the pair $(S, \{X_1,..., X_m\})$, and the set supp S^{dec} as $\operatorname{supp} S^{\text{dec}} = \operatorname{supp} S \cup X_1 \cup \cdots \cup X_m$. We say that p is a plaquette in S^{dec} if $p \subset \operatorname{supp} S^{\text{dec}}$. We say that p is a *simple* plaquette in S^{dec} if p is parallel to S_0 and if p is the only plaquette in S^{dec} that has the projection $\pi(p)$. A plaquette p in S^{dec} is called *exited* if it is not simple or if it is touched by a plaquette in S^{dec} that is not simple.

Let $C_1,..., C_n$ be the connected components of the set of excited plaquettes in S^{dec}, and let $P_i = \pi(C_i)$ be the corresponding projections onto S_0. We then define the *decorated walls* of S^{dec} as the triples $W_i = (\operatorname{supp}_S W_i, \alpha_{W_i}, \mathbb{X}_{W_i})$, where $\operatorname{supp}_S W_i$ is the union of all plaquettes $p \subset S$ with $\pi(p) \subset P_i$, $\mathbb{X}_{W_i}$ is the set of all clusters $X \in \{X_1,..., X_m\}$ that project onto P_i, and α_{W_i} is an assignment of a label $\alpha_{W_i}(c) = \alpha_S(c)$ to each cube c that touches S and projects onto P_i. The set C_i is called the support[7] of W_i, and $\pi(W_i)$ is defined as $\pi(C_i)$. The *flat pieces* $F_1,..., F_k$ of S^{dec}, on the other hand, are defined as the connected components of the set of unexcited plaquettes in S^{dec}. A plaquette p in a decorated wall W (in a flat piece F) is called a boundary plaquette of W (of F), if $\pi(p)$ is connected to $\partial\pi(W)$ (to $\partial\pi(F)$). The height of a plaquette is the height of its center. Finally, we say

[7] Note that $C_i = \operatorname{supp}_S W_i \cup \bigcup \{X \mid X \in \mathbb{X}_{W_i}\}$.

that W is a decorated wall if there exists a decorated interface S^{dec} such that W is a decorated wall of S^{dec}.

Remark. Following [Dob72], we have defined the notion of excited plaquettes in such a way that two boundary plaquettes p_W and p_F of a decorated wall W^{dec} and a flat piece F have the same height if $\pi(p_W^{\text{dec}})$ and $\pi(p_F)$ are connected. Therefore the decorated interface S^{dec} can be reconstructed uniquely from its decorated walls $W_1,..., W_n$. We express this fact by writing $S^{\text{dec}} = S^{\text{dec}}(W_1,..., W_n)$ if S^{dec} is a decorated interface with decorated walls $W_1,..., W_n$.

In order to derive an expansion for the surface tension σ, we need some facts about the geometry of the decorated interface S^{dec} and its decomposition into decorated walls and flat pieces. We call a translation $t_s^{(1)}: V_\infty \to V_\infty$, $t_s^{(1)}(x) = x + (s, 0,..., 0)$ a vertical translation by s. For a set C in V_∞, we use $[C]$ to denote the equivalence class under vertical translations,

$$[C] = \{\tilde{C} \mid \tilde{C} = t_s^{(1)}(C) \text{ for some } s \in \mathbb{Z}\}$$

If S^{dec} is a decorated interface with decorated walls $W_1,..., W_n$, we call the equivalence classes $[W_1],..., [W_n]$ the "floating decorated walls" of S. Given two decorated walls W, $\tilde{W}$, we say that $[W]$ and $[\tilde{W}]$ are *compatible*, if $\pi(W)$ and $\pi(\tilde{W})$ are not connected to each other. With this definition, the floating walls $[W_1],..., [W_n]$ of a decorated interface S are pairwise compatible.

The following lemma is an immediate consequence of the fact that two boundary plaquettes p_W and p_F of a decorated wall W and a flat piece F have the same height if $\pi(p_W)$ and $\pi(p_F)$ are connected, see the remark above. The necessary geometric constructions are identical to those in [Dob72] and are not repeated here.

Lemma 5.2 [Dob72]. Let $d \geqslant 3$, and let $\{[W_1],..., [W_n]\}$ be a set of pairwise compatible floating decorated walls. Then there is exactly one decorated interface $S^{\text{dec}} = S^{\text{dec}}([W_1],..., [W_n])$ with $[W_1],..., [W_n]$ as its floating decorated walls.

Consider now a decorated interface $S^{\text{dec}} = (S, \{X_1,..., X_m\})$ with decorated walls $W_1,..., W_n$. For $W \in \{W_1,..., W_n\}$, let S_W be the pair $S_W = (\text{supp}_S W, \alpha_W)$. By our definition of excited plaquettes and the fact that the interaction terms $\lambda_A \prod_{x \in A} s_x$ in (1.2) are zero once the diameter of A is greater than 1, the weight $\rho(S)$ of the interface S factors into a product

$$\rho(S) = \rho(S_0) \prod_{i=1}^{n} \rho(S_{W_i}) \tag{5.31}$$

where $\rho(S_{W_i})$ obeys the bound

$$|\rho(S_{W_i})| \leqslant e^{-\tau(|\mathrm{supp}_S W_i| - |\pi(\mathrm{supp}_S W_i)|)} \tag{5.32}$$

Here τ is the constant defined in (2.6) and $|\pi(\mathrm{supp}_S W_i)|$ is the $(d-1)$-dimensional area of $\pi(\mathrm{supp}_S W_i)$. Note also that

$$\Delta F(S(W_1,..., W_n)) = \sum_{i=1}^{n} \Delta F(S([W_i])) \tag{5.33}$$

where $S([W_i])$ is the interface which has $[W_i]$ as its only floating wall. Defining the weight $z(W)$ of a decorated wall $W = (\mathrm{supp}_S W, \alpha_W, \mathbb{X}_W)$ as

$$z(W) = \rho(S_W)\, e^{\Delta F(S([W]))} \prod_{X \in \mathbb{X}_W} z(X, S_W) \tag{5.34}$$

we therefore obtain the decomposition

$$\frac{\rho(S)}{\rho(S_o)}\, e^{\Delta F(S)} \prod_{j=1}^{m} z(X_j, S) = \prod_{i=1}^{n} z(W_i) \tag{5.35}$$

for the weight of a decorated interface $S^{\mathrm{dec}} = (S, \{X_1,..., X_m\})$ with decorated walls $W_1,..., W_n$.

Remarks.(i) It is easy to see that $z(W) = z(W')$ if $W = t_s^{(1)}(W')$ for some $s \in \mathbb{Z}$. As a consequence, $z(W)$ is actually well defined on the equivalence class $[W]$,

$$z(W) = z([W]) \tag{5.36}$$

(ii) For a decorated wall $W = (\mathrm{supp}_S W, \alpha_W, \mathbb{X}_W)$, let

$$|W| = |\mathrm{supp}\, W| + \sum_{X \in \mathbb{X}_W} |X| \tag{5.37}$$

Then

$$|z(W)| \leqslant e^{-(\tau - O(1))(|W| - |\pi(W)|)} \leqslant e^{-(\tau - O(1))\, \varepsilon(d)\, |W|} \tag{5.38}$$

where the first inequality follows from (5.18), (5.32) and the fact that $\mathrm{Re}\, \Delta F(S) = 0$ for all interfaces S, and $\varepsilon(d) > 0$.

Combining Lemma 5.2 with the decomposition (5.35) we obtain $\tilde{Z}_{+-}$ as a sum over sets $\{[W_1],..., [W_n]\}$ of pairwise compatible floating decorated walls,

$$\tilde{Z}_{+-}(V_\infty) = \sum_{\{[W_1],..., [W_n]\}} \left(\prod_{i=1}^{n} z([W_i]) \right) \tag{5.39}$$

In the form (5.39), $Z_{+-}(V_\infty)$ is the partition function of a hard-core interacting polymer systems over the flat interface S_0. Its logarithm can therefore be analyzed by the usual Mayer expansion for polymer systems, leading immediately to an expansion for the interface tension σ, see equation (5.1). Note that the Mayer expansion for $\log \tilde{Z}_{+-}(V_\infty)$ is absolutely convergent due to the bound (5.38) and the fact that the number of walls W with $|W| = s$ that are incompatible with a given wall $\tilde{W}$ is bounded by $|\tilde{W}|$ times a geometrical constant to the power s.

6. CLUSTER EXPANSION FOR THE STATES $\langle \cdot \rangle_{+-}$

In this section we derive a cluster expansion for the expectation value $\langle A \rangle_{+-}$, where A is a local observable with support $\mathscr{A} \subset \mathbb{Z}^d_{1/2}$, and use this to prove Theorem 2.2. As before, we take $\operatorname{supp} A = \bigcup_{x \in \mathscr{A}}$. Without loss of generality, we assume that $\operatorname{supp} A$ is a subset of V that does not touch its boundary. At this point, we will not assume, however, that $\operatorname{supp} A$ is a connected set.

It is again convenient to consider unnormalized expectation values; here we define two such quantities,

$$Z_{+-}(A; V) = Z_{+-}(V) \langle A \rangle^V_{+-} \tag{6.1}$$

and

$$\tilde{Z}_{+-}(A; V) = \tilde{Z}_{+-}(V) \langle A \rangle^V_{+-} \tag{6.2}$$

Given a configuration $s_{\bar{\Lambda}}$ contributing to (6.1), we define the interface S and the contours $Y_1,..., Y_N$ corresponding to $s_{\bar{\Lambda}}$ as in Section 5. As in Section 4, we then group all contours Y with $V(Y) \cap \operatorname{supp} A \neq \varnothing$ into a new contour Y_A with $\operatorname{supp} Y_A = \bigcup_{Y \in Y_A} \operatorname{supp} Y$. Resumming the remaining contours, and defining $V_q(S, Y_A)$ as the union of all components of $V \backslash (\operatorname{supp} S \cup \operatorname{supp} Y_A)$ with label q and $V_q^{(0)}(Y_A, S)$ as $V_q^{(0)}(Y_A, S) = V_q(Y_A, S) \backslash \operatorname{supp} A$, $q = \pm 1$, we obtain the representation

$$\begin{aligned} Z_{+-}(A; V) &= \sum_{S,\, Y_A} \rho_A(Y_A, S) \prod_{q = \pm 1} Z_q(V_q^{(0)}(Y_A, S)) \\ &= \sum_S E_V(A \mid S)\, \rho(S) \prod_{q = \pm 1} Z_q(V_q(S)) \end{aligned} \tag{6.3}$$

where

$$\rho_A(Y_A, S) = A(Y_A, S)\, \rho(S) \left(\prod_{Y \in Y_A} \rho(Y) \right) \prod_{q = \pm 1} e^{-e_q |\text{supp}\, A \cap V_q(Y_A, S)|} \tag{6.4}$$

and

$$E_V(A \mid S) = \sum_{Y_A:\, \text{supp}\, Y_A \cap \text{supp}\, S \neq \varnothing} A(Y_A, S) \left(\prod_{Y \in Y_A} \rho(Y) \right) \times \prod_{q = \pm 1} \frac{Z_q(V_q^{(0)}(Y_A, S))}{Z_q(V_q(S))} e^{-e_q |\text{supp}\, A \cap V_q(Y_A, S)|} \tag{6.5}$$

Extracting the factor $\rho(S_o) \sqrt{Z_+(V)\, Z_-(V)}$ from (6.3), we obtain the representation

$$\tilde{Z}_{+-}(A; V) = \sum_X \left(\frac{\rho(S)}{\rho(S_o)} \prod_{q = \pm 1} \frac{Z_q(V_q(S))}{\sqrt{Z_q(V)}} \right) E_V(A \mid S) \tag{6.6}$$

for the modified partition function (6.2).

Remark. Combining (6.2) and. (6.6), the expectation value $\langle A \rangle_{+-}^V$ can be written in the form

$$\langle A \rangle_{+-}^V = \sum_S P_V(S)\, E_V(A \mid S)$$

If the weights $P_V(S)$ were positive, it would be straightforward to control $\langle A \rangle_{+-}^V$: The expansions of the last section yield bounds on the probabilities $P_V(S)$, and the standard cluster expansions yield bounds on the difference between $E_V(A \mid S)$ and $\langle A \rangle_{\pm}^V$. Here, due to the complexity of the weights $P_V(S)$, we have to follow a different route, which we now describe.

In order to analyze the conditional expectation values $E_V(A \mid S)$, we use the methods sketched in Section 4. We again consider unnormalized expectation values,

$$Z(A; V \mid S) = E_V(A \mid S)\, Z_+(V_+(S))\, Z_-(V_-(S)) \tag{6.7}$$

Defining the conditional activities

$$\rho_A(Y_A \mid S) = A(Y_A, S) \left(\prod_{Y \in Y_A} \rho(Y) \right) \prod_{q = \pm 1} e^{-e_q |\text{supp}\, A \cap V_q(Y_A, S)|} \tag{6.8}$$

and

$$K_A(Y_A \mid S) = \rho_A(Y_A \mid S) \prod_{q=\pm 1} \left(e^{e_q |\operatorname{supp} A \cap V_q(S)|} \prod_{m=\pm 1} \frac{Z_m(\operatorname{Int}_m^{(0)} Y_A \cap V_q(S))}{Z_q(\operatorname{Int}_m^{(0)} Y_A \cap V_q(S))} \right) \tag{6.9}$$

we rewrite $Z(A; V \mid S)$ as

$$\begin{aligned} Z(A; V \mid S) &= \sum_{Y_A:\, \operatorname{supp} Y_A \cap \operatorname{supp} S \neq \varnothing} \rho_A(Y_A \mid S) \prod_{q=\pm 1} Z_q(V_q^{(0)}(Y_A, S)) \\ &= \sum_{Y_A:\, \operatorname{supp} Y_A \cap \operatorname{supp} S \neq \varnothing} \rho_A(Y_A \mid S) \left(\prod_{q=\pm 1} Z_q(\operatorname{Ext}^{(0)} Y_A \cap V_q(S) \right) \\ &\quad \times \left(\prod_{q,m=\pm 1} Z_m(\operatorname{Int}_m^{(0)} Y_A \cap V_q(S)) \right) \\ &= \sum_{Y_A:\, \operatorname{supp} Y_A \cap \operatorname{supp} S \neq \varnothing} K_A(Y_A \mid S) \\ &\quad \times \prod_{q=\pm 1} \left(e^{-e_q |\operatorname{supp} A \cap V_q(S)|} Z_q(V_s(S) \backslash (\operatorname{supp} Y_A \cup \operatorname{supp} A)) \right) \end{aligned} \tag{6.10}$$

where in the second step, we first used that

$$\begin{aligned} V_q^{(0)}(Y_A, S) &= (\operatorname{Ext}^{(0)} Y_A \cap V_q(S)) \\ &\quad \dot{\cup} (\operatorname{Int}_q^{(0)} Y_A \cap V_+(S)) \dot{\cup} (\operatorname{Int}_q^{(0)} Y_A \cap V_-(S)) \end{aligned}$$

and then interchanged the indices m and q in the second product. Inserting the representation (3.7) for $Z_q(V_q(S) \backslash (\operatorname{supp} Y_A \cup \operatorname{supp} A))$ into (6.10), we get

$$\begin{aligned} Z(A; V \mid S) &= e^{-e_+ |V_+(S)| - e_- |V_-(S)|} \\ &\quad \times \sum_{Y_A:\, \operatorname{supp} Y_A \cap \operatorname{supp} S \neq \varnothing} K_A(Y_A \mid S) \sum_{\{Y_1, \ldots, Y_n\}} \prod_{k=1}^{n} K(Y_k) \end{aligned} \tag{6.11}$$

Here the second sum goes over sets of non-overlapping contours in $V \backslash \operatorname{supp} S$, such that for all contours $Y \in \{Y_1, \ldots, Y_n\}$, the set $V(Y)$ does not intersect the set $\operatorname{supp} Y_A \cup \operatorname{supp} A$, and

$$K(Y) = \begin{cases} K_+(Y) & \text{if } \operatorname{supp} Y \subset V_+(S) \\ K_-(Y) & \text{if } \operatorname{supp} Y \subset V_-(S) \end{cases} \tag{6.12}$$

Continuing as in Section 4, we get $E_V(A \mid S)$ as a sum over A-clusters X_A in $V\backslash\mathrm{supp}\, S$ that do not touch the boundary of $V\backslash\mathrm{supp}\, S$,

$$E_V(A \mid S) = \sum_{X_A \subset V\backslash\mathrm{supp}\, S} k_A(X_A \mid S) \tag{6.13}$$

where $k_A(X_A \mid S)$ is obtained from (4.8) by replacing $K_A(Y_A \mid S)$ for $K_{q,A}(Y_A)$ and $K(Y_k)$ for $K_q(Y_k)$.

Returning to the expansion (6.6), the weights

$$\frac{\rho(S)}{\rho(S_o)} \frac{Z_+(V_+(S))}{\sqrt{Z_+(V)}} \frac{Z_-(V_-(S))}{\sqrt{Z_-(V)}} = \frac{\rho(S)}{\rho(S_o)} e^{\Delta F(S) + W_V(S)}$$

can be analyzed by the methods of Section 5, leading to the existence of the limit

$$\begin{aligned}\tilde{Z}_{+-}(A; V_\infty) &= \lim_{L_\perp \to \infty} \tilde{Z}_{+-}(A \mid V(L_\perp, L)) \\ &= \sum_S \left(\frac{\rho(S)}{\rho(S_o)} e^{\Delta F(S) + W(S)} \right) E_{V_\infty}(A \mid S) \end{aligned} \tag{6.14}$$

as in Lemma 5.1, and the representation

$$\tilde{Z}_{+-}(A; V_\infty) = \sum_{\{[W_1],\dots,[W_n]\}} \left(\prod_{i=1}^n z([W_i]) \right) E_{V_\infty}(A \mid S) \tag{6.15}$$

where the sum goes over sets $\{[W_1],\dots,[W_n]\}$ of pairwise compatible floating decorated walls and $S = S([W_1],\dots,[W_n])$, as in Eq. (5.39).

Inserting finally the expansion (6.13) for $E_{V_\infty}(A \mid S)$, we get

$$\tilde{Z}_{+-}(A; V_\infty) = \sum_{\{[W_1],\dots,[W_n]\}} \sum_{X_A \subset V_\infty\backslash\mathrm{supp}\, S} k_A(X_A \mid S) \prod_{i=1}^n z([W_i]) \tag{6.16}$$

where $S = S([W_1],\dots,[W_n])$.

At this point, we define $[W] \leftrightarrow X_A$ iff $\pi(W)$ intersects or surrounds the projection of $X_A \cup \mathrm{supp}\, A$. Observing that $k_A(X_A \mid S)$ depends on S only via the walls W with $[W] \leftrightarrow X_A$, we define a decorated A-wall as a pair $W_A = (X_A, \{[W_1],\dots,[W_m]\})$ where X_A is an A-cluster and $\{[W_1],\dots,[W_m]\}$ is a set of pairwise compatible floating decorated walls such that $[W] \leftrightarrow X_A$ for all $[W] \in \{[W_1],\dots,[W_m]\}$. We say that a floating wall $[W]$ and a decorated A-wall $W_A = (X_A, \{[W1],\dots,[W_m]\})$ are compatible if $\{[W],[W_1],\dots,[W_m]\}$ is a set of pairwise compatible floating decorated walls, and if $[W] \nleftrightarrow X_A$ for the A-cluster X_A in W_A. Defining the activity of a decorated A-wall $W_A = (X_A, \{[W_1],..., [W_m]\})$ as

$$z_A(W_A) = k_A(X_A \mid S([W_1],..., [W_m])) \prod_{[W] \in \{[W_1],..., [W_m]\}} z([W]) \tag{6.17}$$

we get

$$\tilde{Z}_{+-}(A; V_\infty) = \sum_{W_A} \sum_{\{[W_1],..., [W_n]\}} z_A(W_A) \left(\prod_{i=1}^{n} z([W_i]) \right) \tag{6.18}$$

where the first sum is over decorated A-walls, and the second is over sets of pairwise compatible floating decorated walls that are all compatible with W_A.

Defining the graph $G(W_A, [W_1],..., [W_n])$ as the graph on $0, 1,..., n$ that has a line between 0 and i, $i = 1,..., n$ whenever $W_A \nleftrightarrow [W_i]$, and a line between i and j ($i, j = 1,..., n$) whenever $[W_i]$ and $[W_j]$ are not compatible, the compatibility constraint on the right hand side of (6.18) can be implemented by inserting a factor $G(W_A, [W_1],..., [W_n])$ which is zero whenever $G(W_A, [W_1],..., [W_n])$ has less than $n+1$ components. In the same way as the representation (4.6) gives the cluster expansion (4.7), our representation (6.18) now gives the cluster expansion

$$\langle A \rangle_{+-}^{V_\infty} = \sum_{n=0}^{\infty} \sum_{W_A} \sum_{[W_1],..., [W_n]} \frac{\phi_c(W_A, [W_1],..., [W_n])}{n!} z_A(W_A) \times \left(\prod_{i=1}^{n} z([W_i]) \right) \tag{6.19}$$

where $\phi_c(WA), [W_1],..., [W_n])$ is obtained from $G(W_A, [W_1],..., [W_n])$ via (3.12).

Remark. As a consequence of the bounds (4.12) and (5.38), we have

$$|z_A(W_A)| \leqslant \|A\| \, e^{O(e^{-\tau}) \, |\mathrm{supp}\, A|} e^{-(\tau - O(1))(|W_A| - |\pi(W_A)|)} \tag{6.20}$$

where the size $|W_A|$ of a decorated A-wall $W_A = (X_A, \{[W_1],..., [W_m]\})$ is defined as

$$|W_A| = |X_A| + \sum_{i=1}^{m} |W_i| \tag{6.21}$$

with $|X_A|$ given by (4.11) and $|W_i|$ given by (5.37). Using the bound (6.20), one easily shows absolute convergence of the expansion (6.19) uniformly in L (recall that $V_\infty = V_\infty(L)$, see (5.2d)).

Theorem 2.2 (i), and the exponential clustering for the directions parallel to S_0 immediately follow from the convergence of the expansion (6.19). we are therefore left with the proof of Theorem 2.2 (ii), and the proof of exponential clustering in the direction orthogonal to S_0, i.e. the proof of (2.19) for translations $t_s^{(1)}$ in the 1-direction. In order to prove these, we slightly modify our expansion. Starting with the proof of Theorem 2.2 (ii), assume that A is an observable with $\operatorname{supp} A \subset V_+(S_0)$. We define

$$\tilde{A} = A - \langle A \rangle_+^{V_\infty} \tag{6.22}$$

so that

$$\langle A \rangle_{+-}^{V_\infty} - \langle A \rangle_+^{V_\infty} = \langle \tilde{A} \rangle_{+-}^{V_\infty} \tag{6.23}$$

Combining the expansion (4.9) for $\langle A \rangle_+^{V_\infty}$ with the expansion (6.13), we then get

$$\begin{aligned} E_{V_\infty}(\tilde{A} \mid S) &= E_{V_\infty}(A \mid S) - \langle A \rangle_+^{V_\infty} \\ &= \sum_{X_A \subset V_\infty \setminus \operatorname{supp} S} k_A(X_A \mid S) - \sum_{X_A \subset V_\infty} k_{+,A}(X_A) \end{aligned} \tag{6.24}$$

Observing that if $(X_A \cup \operatorname{supp} A) \subset V_+(S)$, then also $\operatorname{Int}^{(0)} X_A \subset V_+(S)$, which together imply $k_A(X_A \mid S) = k_{+,A}(X_A)$, we get the representation

$$E_{V_\infty}(\tilde{A} \mid S) = \sum_{X_A : (X_A \cup \operatorname{supp} A) \not\subset V_+(S)} \tilde{k}_A(X_A|S) \tag{6.25}$$

where $\tilde{k}_A(X_A \mid S) = k_A(X_A \mid S) - k_{+,A}(X_A)$ if $X_A \subset V_\infty \setminus \operatorname{supp} S$ and $\tilde{k}_A(X_A \mid S) = -k_{+,A}(X_A)$ if $X_A \not\subset V_\infty \setminus \operatorname{supp} S$.

Next we decompose $\langle \tilde{A} \rangle_{+-}^{V_\infty}$ into two terms

$$\langle \tilde{A} \rangle_{+-}^{V_\infty} = E_{+-}^{V_\infty,(1)}(\tilde{A}) + E_{+-}^{V_\infty,(2)}(\tilde{A}) \tag{6.26}$$

where

$$E_{+-}^{V_\infty,(i)}(\tilde{A}) = \frac{\tilde{Z}_{+-}^{(i)}(\tilde{A}; V_\infty)}{\tilde{Z}_{+-}(V_\infty)} \tag{6.27}$$

with

$$\tilde{Z}^{(1)}_{+-}(\tilde{A}; V_\infty) = \sum_{S:\, \operatorname{supp} A \subset V_+(S)} \left(\frac{\rho(S)}{\rho(S_o)} e^{\Delta F(S) + W(S)} \right) E_{V_\infty}(A \mid S) \tag{6.28}$$

and

$$\tilde{Z}^{(2)}_{+-}(\tilde{A}; V) = \sum_{S:\, \operatorname{supp} A \not\subset V_+(S)} \left(\frac{\rho(S)}{\rho(S_o)} e^{\Delta F(S) + W(S)} \right) E_{V_\infty}(A \mid S) \tag{6.29}$$

Inserting the expansion (6.25) into (6.28), we get a sum over A-clusters X_A that contain at least one component X with $X \cap \operatorname{supp} S \neq \emptyset$ and $V(X) \cap \operatorname{supp} A \neq \emptyset$, where the latter condition follows directly from the definition of A-clusters. The expansion (6.19) for $E^{V_\infty,(1)}_{+-}(\tilde{A})$ therefore only contains decorated A-walls $W_A = (X_A, \{[W_1],..., [W_m]\})$ for which X_A contains a component that is connected to the interface $S([W_1],..., [W_m])$. As a consequence, the sum over W_A only goes over decorated A-walls W_A with $|W_A| - |\pi(W_A)| \geqslant \operatorname{dist}(S_0, \operatorname{supp} A)$, which by (6.20) leads to the bound

$$|E^{V_\infty,(1)}_{+-}(\tilde{A})| \leqslant \|A\| \, e^{O(e^{-\tau}) |\operatorname{supp} A|} e^{-(\tau - O(1)) \operatorname{dist}(S_0, \operatorname{supp} A)} \tag{6.30}$$

In order to estimate the second term in (6.26), we note that the condition $\operatorname{supp} A \not\subset V_+(S)$ in (6.29) implies that the expansion (6.19) for $E^{V_\infty,(2)}_{+-}(\tilde{A})$ only contains decorated A-walls $W_A = (X_A, \{[W_1],..., [W_m]\})$ with $\operatorname{supp} A \not\subset V_+(S([W_1],..., [W_m]))$. As a consequence, the corresponding sum over W_A only involves decorated A-walls $W_A = (X_A, \{[W_1],..., [W_m]\})$ with

$$\sum_{i=1}^{m} (|W_i| - |\pi(W_i)|) \geqslant \operatorname{dist}(S_0, \operatorname{supp} A) \tag{6.31}$$

which by (6.20) and (6.21) leads to the bound

$$|E^{V_\infty,(2)}_{+-}(\tilde{A}) \leqslant \|A\| \, e^{O(e^{-\tau}) |\operatorname{supp} A|} e^{-(\tau - O(1)) \operatorname{dist}(S_0, \operatorname{supp} A)} \tag{6.32}$$

The bounds (6.30) and (6.32) imply (2.17). The bound (2.18) is proven in the same way.

We are left with the proof of Theorem 2.2 for translations t_x in the direction orthogonal to S_0, $t_x = t_s^{(1)}$. Assuming without loss of generality that s is a positive integer that is chosen large enough to ensure that $\operatorname{supp} t_s^{(1)}(B) \subset V_+(S_0)$, we bound

$$\begin{aligned} &|\langle A t_s^{(1)}(B)\rangle_{+-}^{V_\infty} - \langle A\rangle_{+-}^{V_\infty} \langle t_s^{(1)}(B)\rangle_{+-}^{V_\infty}| \\ &\quad \leqslant |\langle A t_s^{(1)}(B)\rangle_{+-}^{V_\infty} - \langle A\rangle_{+-}^{V_\infty} \langle B\rangle_{+}^{V_\infty}| \\ &\qquad + |\langle A\rangle_{+-}^{V_\infty} (\langle B\rangle_{+}^{V_\infty} - \langle t_s^{(1)}(B)\rangle_{+-}^{V_\infty})| \end{aligned} \tag{6.33}$$

The second term decays exponentially in s due to the bounds (6.30) and (6.32) and the fact that $|\langle A\rangle_{+-}^{V_\infty}| \leqslant \|A\| \, e^{O(e^{-\tau}) \, |\operatorname{supp} A|}$:

$$|\langle A\rangle_{+-}^{V_\infty} (\langle B\rangle_{+}^{V_\infty} - \langle t_s^{(1)}(B)\rangle_{+-}^{V_\infty})| \leqslant C_{AB} e^{-(\tau - O(1)) \operatorname{dist}(S_0, \operatorname{supp} t_s^{(1)}(B))} \tag{6.34}$$

where $C_{AB} = \|A\| \, e^{O(e^{-\tau}) \, |\operatorname{supp} A|} \, \|B\| \, e^{O(e^{-\tau}) \, |\operatorname{supp} B|}$. Therefore we need only bound

$$\langle A\tilde{B}\rangle_{+-}^{V_\infty} = \langle A t_s^{(1)}(B)\rangle_{+-}^{V_\infty} - \langle A\rangle_{+-}^{V_\infty} \langle B\rangle_{+}^{V_\infty} \tag{6.35}$$

where

$$\tilde{B} = t_s^{(1)}(B) - \langle B\rangle_{+}^{V_\infty} \tag{6.36}$$

Again, we first analyze the conditional expectations $E_{V_\infty}(A\tilde{B} \mid S)$, which we rewrite as

$$\begin{aligned} E_{V_\infty}(A\tilde{B} \mid S) &= E_{V_\infty}(A t_s^{(1)}(B) \mid S) - E_{V_\infty}(A \mid S) \langle B\rangle_{+}^{V_\infty} \\ &= E_{V_\infty}(A; t_s^{(1)}(B) \mid S) + E_{V_\infty}(A \mid S) \, E_{V_\infty}(\tilde{B} \mid S) \end{aligned} \tag{6.37}$$

where $E_{V_\infty}(A; t^{(1)}(B) \mid S)$ denotes the "truncated conditional expectation:"

$$E_{V_\infty}(A; t_s^{(1)}(B) \mid S) = E_{V_\infty}(A t_s^{(1)}(B) \mid S) - E_{V_\infty}(A \mid S) \, E_{V_\infty}(t_s^{(1)}(B) \mid S) \tag{6.38}$$

It follows from the decomposition (6.37) that

$$\langle A\tilde{B}\rangle_{+-}^{V_\infty} = E_{+-}^{V_\infty, (a)}(A, \tilde{B}) + E_{+-}^{V_\infty, (b)}(A, \tilde{B}) + E_{+-}^{V_\infty, (c)}(A, \tilde{B}) \tag{6.39}$$

where

$$E_{+-}^{V_\infty, (i)}(A, \tilde{B}) = \frac{\tilde{Z}_{+-}^{(i)}(A; \tilde{B}; V_\infty)}{\tilde{Z}_{+-}(V_\infty)} \tag{6.40}$$

with

$$\tilde{Z}^{(a)}_{+-}(A, \tilde{B}; V_\infty) = \sum_S \left(\frac{\rho(S)}{\rho(S_o)} e^{\Delta F(S) + W(S)} \right) E_{V_\infty}(A; t_s^{(1)}(B) \mid S) \tag{6.41}$$

$$\tilde{Z}^{(b)}_{+-}(A, \tilde{B}; V) = \sum_{S:\, \operatorname{supp} \tilde{B} \subset V_+(S)} \left(\frac{\rho(S)}{\rho(S_o)} e^{\Delta F(S) + W(S)} \right) E_{V_\infty}(A \mid S)\, E_{V_\infty}(\tilde{B} \mid S) \tag{6.42}$$

and

$$\tilde{Z}^{(c)}_{+-}(A, \tilde{B}; V) = \sum_{S:\, \operatorname{supp} \tilde{B} \not\subset V_+(S)} \left(\frac{\rho(S)}{\rho(S_o)} e^{\Delta F(S) + W(S)} \right) E_{V_\infty}(A \mid S)\, E_{V_\infty}(\tilde{B} \mid S) \tag{6.43}$$

The rest is straightforward: Defining an $A\tilde{B}$-cluster $X_{A\tilde{B}}$ as a union of finitely many clusters X with $V(X) \cap \operatorname{supp} A \cup \operatorname{supp} t_s^{(1)}(B)) \neq \emptyset$, and a decorated $A\tilde{B}$-wall as a pair $W_{A\tilde{B}} = (X_{A\tilde{B}}, \{[W_1],..., [W_m]\}$ where $X_{A\tilde{B}}$ is an $A\tilde{B}$-cluster and $\{[W_1],..., [W_m]\}$ is a set of pairwise compatible floating decorated walls such that $[W] \leftrightarrow X_{A\tilde{B}}$ for all $[W] \in \{[W_1],..., [W_m]\}$, we proceed as in the derivation of (6.19) to expand $E^{V_\infty, (i)}_{+-}(A, \tilde{B})$ as a sum involving decorated $A\tilde{B}$-walls.

Starting with the expansion for $E^{V_\infty, (a)}_{+-}(A, \tilde{B})$, we note that the cluster expansion (6.13) for the truncated expectation values (6.38) gives a sum over $A\tilde{B}$-clusters $X_{A\tilde{B}}$ that have at least one component X with $X \cap \operatorname{supp} A \neq \emptyset$ and $X \cap \operatorname{supp} t_s^{(1)}(B) \neq \emptyset$. As a consequence, the expansion (6.19) for $E^{V_\infty, (a)}_{+-}(A, \tilde{B})$ only involves decorated $A\tilde{B}$-walls with $|W_{A\tilde{B}}| - |\pi(W_{A\tilde{B}}| \geqslant \operatorname{dist}(\operatorname{supp} A, \operatorname{supp} t_s^{(1)}(B))$, leading to an upper bound of the form

$$|E^{V_\infty, (a)}_{+-}(A, \tilde{B})| \leqslant C_{AB} e^{-(\tau - O(1)) \operatorname{dist}(\operatorname{supp} A,\, \operatorname{supp} t_s^{(1)}(B))} \tag{6.44}$$

In order to estimate the second and third term in (6.39), we use the expansion (6.13) for $E_{V_\infty}(A \mid S)$ and the expansion (6.25) for $E_{V_\infty}(\tilde{B} \mid S)$. Inserted into (6.42), this gives an expansion of $A\tilde{B}$-clusters that are unions of arbitrary A-clusters with $\tilde{B}$-clusters that contain at least one component X with $X \cap \operatorname{supp} S \neq \emptyset$ and $V(X) \cap \operatorname{supp} t_s^{(1)}(B) \neq \emptyset$. Continuing as in the above proof of (6.30), this leads to a bound of the form

$$|E^{V_\infty, (b)}_{+-}(A, \tilde{B})| \leqslant C_{AB} e^{-(\tau - O(1)) \operatorname{dist}(S_0,\, \operatorname{sup} t_s^{(1)}(B))} \tag{6.45}$$

Finally, the expansion for $E_{+-}^{V_\infty,(c)}(A, \tilde{B})$ only involves interfaces S with $\operatorname{supp} \tilde{B} \not\subset V_+(S)$, and hence only decorated $A\tilde{B}$-walls $W_{A\tilde{B}} = (X_{A\tilde{B}}, \{[W_1],..., [W_m]\})$ with

$$\sum_{i=1}^{m} (|W_i| - |\pi(W_i)|) \geqslant \operatorname{dist}(S_0, \operatorname{supp} t_s^{(1)}(B)) \tag{6.46}$$

leading to the bound

$$|E_{+-}^{V_\infty,(c)}(A, \tilde{B})| \leqslant C_{AB} e^{-(\tau - O(1)) \operatorname{dist}(S_0, \operatorname{supp} t_s^{(1)}(B))} \tag{6.47}$$

Equations (6.33)–(6.35) combined with (6.39), (6.44), (6.45) and (6.47) give the desired exponential clustering in the directions orthogonal to S_0, and complete the proof of Theorem 2.2.

ACKNOWLEDGMENTS

The work of C. B. was partly supported by the Commission of the European Union under contract CHRX-CTB-0411; the work of J. T. C. was partly supported by NSF Grant DMS-9403842.

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